Lie Groups and Lie Algebras

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1 Matrix Lie Groups

Throughout this section, we shall let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. Let K be a field. We define the **general linear group of degree n** over K, denoted $\mathrm{GL}(n, K)$ to be the group of all $n \times n$ invertible matrices with group operation given by matrix multiplication.

Remark. $\mathrm{GL}(n, K)$ is an $n \times n$ dimensional vector space over K. Given a norm on $|| \cdot ||$ on K, we can define a norm on $\mathrm{GL}(n, K)$, say $|| \cdot ||_M$ as follows:

$$||A||_M = \sum_{i=1}^{n} \sum_{j=1}^{n} ||A_{ij}||$$

Recall that all norms on finite dimensional vector spaces over locally compact fields (such as $\mathbb{F}$) are equivalent. Hence over a locally compact field K, all norms on $\mathrm{GL}(n, K)$ are equivalent.

Definition 1.2. We define a **matrix Lie group** to be a topologically closed subgroup of $\mathrm{GL}(n, \mathbb{F})$ for some $n \geq 1$.

Example 1.3. $\mathrm{GL}(n, \mathbb{F})$ is a matrix Lie group since the whole group is closed in itself.

Example 1.4. Let $\mathrm{SL}(n, \mathbb{F}) = \{ A \in \mathrm{GL}(n, \mathbb{F}) \mid \det A = 1 \}$. Then $\mathrm{SL}(n, \mathbb{F})$ is a matrix Lie group. Indeed, the function

$$\begin{aligned} \det : \mathrm{GL}(n, \mathbb{F}) &\to \mathbb{F} \\ A &\mapsto \det A \end{aligned}$$

is continuous. Then $\mathrm{SL}(n, \mathbb{F}) = \det^{-1}(\{ 1 \})$. Recall that the inverse image of a closed set under a continuous map is closed and so $\mathrm{SL}(n, \mathbb{F})$ is closed.

Example 1.5. Let $\mathrm{O}(n, \mathbb{F}) = \{ A \in \mathrm{GL}(n, \mathbb{F}) \mid A^t A = \mathbb{1}_n \}$. Then $\mathrm{O}(n, \mathbb{F})$ is a matrix Lie group. Indeed, the function

$$\begin{aligned} \phi : \mathrm{GL}(n, \mathbb{F}) &\to \mathrm{GL}(n, \mathbb{F}) \\ A &\mapsto A^t A \end{aligned}$$

is continuous. Then $\mathrm{O}(n, \mathbb{F}) = \phi^{-1}(\{ \mathbb{1}_n \})$. $\{ \mathbb{1}_n \}$ is clearly closed whence $\mathrm{O}(n, \mathbb{F})$ is closed.

Example 1.6. Let $\mathrm{SO}(n,\mathbb{F}) = \{ A \in \mathrm{GL}(n,\mathbb{F}) \mid A^t A = \mathbb{1}_n, \det A = 1 \}$. Then $\mathrm{SO}(n,\mathbb{F})$ is a matrix Lie group. Indeed, $\mathrm{SO}(n,\mathbb{F}) = \det^{-1}(\{ 1 \}) \cap \phi^{-1}(\{ \mathbb{1}_n \})$. This is the intersection of two closed sets so it is again a closed set.

Example 1.7. The same argumentation as the previous examples applies to the groups $\mathrm{U}(n,\mathbb{F}) = \{ A \in \mathrm{GL}(n,\mathbb{F}) \mid A^\dagger A = \mathbb{1} \}$ and $\mathrm{SU}(n,\mathbb{F}) = \{ A \in \mathrm{GL}(n,\mathbb{F}) \mid A^\dagger A = \mathbb{1}, \det A = 1 \}$.

Definition 1.8. Let G_1 and G_2 be matrix Lie groups over $\mathbb{F}$. A **Lie group homomorphism** $\varphi : G_1 \to G_2$ is a continuous group homomorphism from G_1 to G_2.

Definition 1.9. Let K be a field and $(V, ||\cdot||)$ a normed K-vector space. We say that a subset $W \subseteq V$ is **compact** if W is closed and bounded with respect to $||\cdot||$.

Example 1.10. $\mathrm{O}(n,\mathbb{F}), \mathrm{SO}(n,\mathbb{F}), \mathrm{U}(n,\mathbb{F}), \mathrm{SU}(n,\mathbb{F})$ are all compact. Indeed, consider the case of $\mathrm{O}(n,\mathbb{F})$. Let $A \in \mathrm{O}(n,\mathbb{F})$. Then $A^t A = \mathbb{1}_n$ means that

$$\sum_{k=1}^{n} a_{ik}^2 = 1$$

for all $1 \leq i \leq n$. Hence $||a_{ik}|| \leq 1$ for all $1 \leq i \leq k \leq n$. It then follows that $||A|| \leq n^2$ whence A is bounded. Since $\mathrm{O}(n,\mathbb{F})$ was shown to be closed, we see that $\mathrm{O}(n,\mathbb{F})$ is compact. The same argumentation holds for the other cases.

$\mathrm{SL}(n,\mathbb{F})$ (for $n \geq 2$) and $\mathrm{GL}(n,\mathbb{F})$ are not compact since they contain matrices whose norms grow arbitrarly large. For example, in the $\mathrm{SL}(n,\mathbb{F})$ case, let $\lambda \neq 0$ and consider the diagonal matrix whose entries are all 1s except for a λ and λ^{-1}. Then this matrix is clearly in $\mathrm{SL}(n,\mathbb{F})$ and its norm grows arbitrarily large as $\lambda \to \infty$.

Definition 1.11. Let G be a matrix Lie group. We say that G is **path-connected** if for all $A, A' \in G$, there exists a continuous mapping called a **path** $p : [0,1] \to G$ such that $p(0) = A'$ and $p(1) = A$.

Proposition 1.12. $\mathrm{U}(n,\mathbb{F})$ *is path-connected.*

Proof. By elementary linear algebra, every matrix $V \in \mathrm{U}(n,\mathbb{F})$ is unitarily diagonalisable. Hence there exists a unitary matrix $\overline{V} \in \mathrm{U}(n,\mathbb{F})$ and $\theta_j \in \mathbb{R}$ such that

$$V = \overline{V}\,\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})\overline{V}^{-1}$$

Now define

$$V(t) = \overline{V}\,\mathrm{diag}(e^{i\theta_1(1-t)}, \dots, e^{i\theta_n(1-t)})\overline{V}^{-1}$$

then $V(t)$ is a continuous function satisfying $V(0) = V$ and $V(1) = \mathbb{1}_n$. We see that any unitary matrix is path-connected to the identity element whence $\mathrm{U}(n,\mathbb{F})$ is path-connected. □

Proposition 1.13. $\mathrm{O}(n,\mathbb{F})$ *is not path-connected.*

Proof. Let $A, A' \in \mathrm{O}(n,\mathbb{F})$. Then $\det A$ and $\det A'$ are ± 1. Recall that det is a continuous function on $\mathrm{O}(n,\mathbb{F})$. By elementary topology, the continuous image of a path-connected space is path-connected. Hence if $\mathrm{O}(n,\mathbb{F})$ were to be path connected, so would $\det(\mathrm{O}(n,\mathbb{F})) = \{ -1, 1 \}$. This is clearly not path-connected so $\mathrm{O}(n,\mathbb{F})$ cannot be path-connected. □

Proposition 1.14. *Let G be a matrix Lie group. The connected component of the identity (in other words, all elements of G that are path-connected to $\mathbb{1}_n$) is a subgroup of G.*

Proof. Let $A, B \in G$ and suppose that $A(t)$ and $B(t)$ are paths in G from $\mathbb{1}_n$ to A and B respectively. Then $A(t)B(t)$ is a path from $\mathbb{1}_n$ to AB and $A(t)^{-1}$ is a path from $\mathbb{1}_n$ to A^{-1}. Hence if A and B are in the connected component of $\mathbb{1}_n$, so is AB and A^{-1} whence the connected component of $\mathbb{1}_n$ is a subgroup of G. □

Example 1.15. $\mathrm{SO}(n, \mathbb{F})$ is the connected component of the identity in $\mathrm{O}(n, \mathbb{F})$.

Definition 1.16. Let G be a matrix Lie group. We say that G is **simply connected** if every closed path can be deformed continuously to a constant path.

Example 1.17. We have $\mathrm{SU}(2, \mathbb{F}) \cong S^3$ and is thus simply-connected since loops on spheres can be continuously shrunk to points.

2 Matrix Exponential

Proposition 2.1. *Let $A \in \mathrm{Mat}(n, \mathbb{F})$. Then*

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is absolutely convergent and satisfies

1. $\exp(0) = \mathbb{1}_n$
2. *If $AB = BA$ then* $\exp(A + B) = \exp(A)\exp(B)$
3. $\exp(A)^{-1} = \exp(-A)$
4. *If C is invertible then* $C\exp(A)C^{-1} = \exp(CAC^{-1})$

Proof. We have that

$$||\exp(A)|| \leq \sum_{n=0}^{\infty} \frac{1}{n!}(||A||) < \infty$$

so $\exp(A)$ converges absolutely.

Part 1: This is clear upon inserting the zero matrix into the definition of the matrix exponential.

Part 2: Let $A, B \in \mathrm{Mat}(n, \mathbb{F})$ be such that $AB = BA$. Then by the Binomial Theorem, we have

$$\begin{aligned}
\exp(A+B) &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} A^k \frac{1}{(n-k)!} B^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{n=m+k} \frac{1}{k!} A^k \frac{1}{m!} B^m \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} A^n \frac{1}{m!} B^m \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} B^m \right) \\
&= \exp(A) \exp(B)
\end{aligned}$$

Part 3: Let $A \in \mathrm{Mat}(n, \mathbb{F})$. Clearly, A commutes with $-A$ so by Part 2, we have $\exp(A)\exp(-A) = \exp(A - A) = \mathbb{1}_n$. Hence, $\exp(A)^{-1} = \exp(-A)$.

Part 4: This follows immediately upon left-multiplying and right-multiplying $\exp(A)$ by C and C^{-1} respectively.

□

Proposition 2.2. *Let $A \in \mathrm{Mat}(n, \mathbb{C})$. Then*

$$\det e^A = e^{\mathrm{Tr}\, A}$$

Proof. Since $\mathbb{C}$ is algebraically closed, A is diagonalisable so we can write $A = PDP^{-1}$ for some diagonal matrix $D \in \mathrm{Mat}(n, \mathbb{C})$ with diagonal elements $\lambda_1, \ldots, \lambda_n$ and some invertible $P \in \mathrm{Mat}(n, \mathbb{C})$. Then $e^A = Pe^D P^{-1}$

$$\det(e^A) = \det(P)\det(e^D)\det(P^{-1}) = \det(e^D) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\mathrm{Tr}\, D} = e^{\mathrm{Tr}\, A}$$

Now, the diagonal matrices are dense in $\mathrm{Mat}(n, \mathbb{C})$ and det and e^X are continuous functions so the equality must hold for all of $\mathrm{Mat}(n, \mathbb{C})$ by continuity. □

Proposition 2.3. *Let $A \in \mathrm{Mat}(n, \mathbb{F})$. Then the map $t \mapsto \exp(tA)$ is a smooth curve through $\mathbb{1}_n$ in $\mathrm{Mat}(n, \mathbb{F})$ and*

$$\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} A$$

In particular,

$$\left. \frac{d}{dt} e^{tA} \right|_{t=0} = A$$

Proof. Since $\exp(A)$ is absolutely convergent, differentiation with respect to a parameter is possible. The proof then follows as in the case of real numbers. $\square$

Proposition 2.4. *Let $A \in \mathrm{Mat}(n, \mathbb{C})$ be such that $||A - \mathbb{1}_n|| < 1$. Then*

$$\log(A) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m}(A - \mathbb{1}_n)^m$$

is absolutely convergent and

1. $\exp(\log(A)) = A$
2. $\log(\exp(B)) = B$ *if* $||B|| \leq \log 2$

Proof. The series only exists when $||A - \mathbb{1}_n|| < 1$. The two properties follow in the same way as for the real numbers (infact, they hold in any formal power series ring over a field such as $\mathbb{R}$). The $\log 2$ condition is necessary to ensure that $\exp(B)$ is in the radius of convergence of log. $\square$

Proposition 2.5 (Trotter Product Formulae)**.** *Let $A, B \in \mathrm{Mat}(n, \mathbb{F})$. Then we have the following two formulas:*

$$\exp(A + B) = \lim_{n\to\infty} \left[\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right]^n$$
$$\exp([A, B]) = \lim_{n\to\infty} \left[\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\exp\left(\frac{-A}{n}\right)\exp\left(\frac{-B}{n}\right)\right]^{n^2}$$

where $[A, B]$ is the commutator $AB - BA$.

Proof. Omitted. $\square$

Theorem 2.6 (Baker-Campbell-Hausdorff Formula)**.** *Let $X, Y \in \mathrm{Mat}(n, \mathbb{C})$ such that $||X|| + ||Y|| < \log 2$. Then $\exp(X)\exp(Y) = \exp(Z)$ where*

$$Z = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i + s_i > 0 \\ 1 \leq i \leq n}} \frac{X^{r_1}Y^{s_1} \ldots X^{r_n}Y^{s_n}}{r_1!s_1! \cdots r_n!s_n!}$$

where $X^{r_n}Y^{s_n} \ldots X^{r_n}Y^{s_n} = \underbrace{[X, [X, \ldots, [X,}_{r_1} \underbrace{[Y, [Y, \ldots [Y,}_{s_1} \underbrace{\ldots [X, [X, \ldots [X,}_{r_n} \underbrace{[Y, [Y, \ldots Y]]}_{s_n} \ldots]]$

Proof. Omitted. $\square$

Remark. The first few terms of the BCH Formula are

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \ldots$$

3 Lie Algebras of Matrix Lie Groups

Throughout this section, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 3.1. Let $\mathbb{F}$ be a field. A **Lie algebra** is an $\mathbb{F}$-vector space L equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ called the **Lie bracket** satisfying $[X,Y] = -[Y,X]$ for all $X, Y \in L$ and the Jacobi identity

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

for all $X, Y, Z \in L$.

Proposition 3.2. *Let G be a matrix Lie group over $\mathbb{F}$. Then*

$$\mathrm{Lie}(G) = \{\, X \in \mathrm{Mat}(n,\mathbb{F}) \mid \exp(tX) \in G \;\forall\, t \in \mathbb{R} \,\}$$

is a Lie algebra over $\mathbb{R}$. We shall call this the Lie algebra of G.

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G)$. We first show that $\mathfrak{g}$ is a $\mathbb{R}$-vector space. Let $X \in \mathfrak{g}$ and $s \in \mathbb{R}$. Then

$$X \in \mathfrak{g} \implies e^{stX} \in G \;\forall\, t \in \mathbb{R} \implies e^{t(sX)} \in G \;\forall\, t \in \mathbb{R} \implies sX \in \mathfrak{g}$$

Now let $X, Y \in \mathfrak{g}$. Then by the Trotter Product Formula, we have

$$e^{t(X+Y)} = \lim_{n\to\infty} \left[\exp\left(\frac{tX}{n}\right) \exp\left(\frac{tY}{n}\right) \right]^n$$

Now, $\exp(tX/n), \exp(tY/n) \in G$ for all n and hence so is their product. Since G is closed, the above limit is in G and so $\exp(t(X+Y)) \in G$ for all $t \in \mathbb{R}$ whence $X + Y \in \mathfrak{g}$.

We now show that $\mathfrak{g}$ possesses a Lie bracket. We claim that the matrix commutator $[A,B] = AB - BA$ is such a commutator. It is clearly $\mathbb{R}$-linear, anti-symmetric and satisfies the Jacobi identity. It suffices to show that if $X, Y \in \mathfrak{g}$ then $[X,Y] \in \mathfrak{g}$. We have that

$$\exp(t[A,B]) = \exp([tA,tB]) = \lim_{n\to\infty} \left[\exp\left(\frac{tA}{n}\right) \exp\left(\frac{tB}{n}\right) \exp\left(\frac{-tA}{n}\right) \exp\left(\frac{-tB}{n}\right) \right]^{n^2}$$

By analysis similar to the above, this limit is contained in G and so $[A,B] \in \mathfrak{g}$. □

Example 3.3. We have an isomorphism of groups $\mathbb{R}_{>0} \cong e^{\mathbb{R}}$ so $\mathrm{Lie}(\mathbb{R}_{>0}) = \mathbb{R}$. Similarly, $\mathrm{U}(1,\mathbb{C}) \cong e^{i\mathbb{R}}$ so $\mathrm{Lie}(\mathrm{U}(1,\mathbb{C})) = i\mathbb{R}$.

Example 3.4. Consider the matrix Lie group $G = \mathrm{GL}(n,\mathbb{F})$. Then for any $X \in \mathrm{Mat}(n,\mathbb{F})$, $X \in \mathfrak{g} = \mathrm{Lie}(G)$ if and only if e^{tX} is invertible for all $t \in \mathbb{R}$. This is clearly true so $\mathfrak{g} = \mathrm{Mat}(n,\mathbb{F})$.

Example 3.5. Consider the matrix Lie group $G = \mathrm{SL}(n,\mathbb{F})$. Then for any $X \in \mathrm{Mat}(n,\mathbb{F})$, $X \in \mathfrak{g} = \mathrm{Lie}(G)$ if and only if e^{tX} is invertible and $\det e^{tX} = 1$. This matrix is clearly invertible. We have that

$$1 = \det e^{tX} = e^{\mathrm{Tr}\, tX} = e^{t\,\mathrm{Tr}\, X}$$

which is true for all $t \in \mathbb{R}$ if and only if $\mathrm{Tr}\, X = 0$. Hence, $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{F}) = \{\, A \in \mathrm{Mat}(n,\mathbb{F}) \mid \mathrm{Tr}\, A = 0 \,\}$.

Example 3.6. Consider the matrix Lie group $G = \mathrm{O}(n,\mathbb{F})$. Then for any $X \in \mathrm{Mat}(n,\mathbb{F})$, $X \in \mathfrak{g} = \mathrm{Lie}(G)$ if and only if $e^{tX}(e^{tX})^t = \mathbb{1}_n$ for all $t \in \mathbb{R}$. Expanding this out we have

$$(\mathbb{1}_n + tX + \mathcal{O}(t^2))(\mathbb{1}_n + tX^t + \mathcal{O}(t^2)) = \mathbb{1}_n$$

whence $X = -X^t$. It then follows that

$$(e^{tx})^t = e^{tX^t} = e^{-tX} = (e^{tX})^{-1}$$

and so $e^{tX} \in \mathrm{O}(n, \mathbb{F})$. We therefore have that

$$\mathfrak{g} = \mathfrak{o}(n, \mathbb{F}) = \{ X \in \mathrm{Mat}(n, \mathbb{F}) \mid X = -X^t \}$$

Furthermore, note that $X = -X^t$ implies that $\mathrm{Tr}\, X = 0$ and so $\det(e^{tX}) = 1$. It then follows that $e^{tX} \in \mathrm{SO}(n, \mathbb{F})$ and so $\mathrm{Lie}(\mathrm{SO}(n, \mathbb{F})) = \mathfrak{g}$.

Example 3.7. Similarly to the previous example, we can show that

$$\mathrm{Lie}(\mathrm{U}(n, \mathbb{F})) = \{ X \in \mathrm{Mat}(n, \mathbb{F}) \mid X = -X^\dagger \}$$

However, $X = -X^\dagger$ does not guarantee that $\mathrm{Tr}\, X = 0$ so

$$\mathrm{Lie}(\mathrm{SU}(n, \mathbb{F})) = \{ X \in \mathrm{Mat}(n, \mathbb{F}) \mid X = -X^\dagger, \mathrm{Tr}\, X = 0 \}$$

Proposition 3.8. *Let G be a matrix Lie group and $X \in \mathrm{Lie}(G)$. Then given any $A \in G$ we have $\mathrm{Ad}_A(X) = AXA^{-1} \in \mathrm{Lie}(G)$. This is referred to as the **adjoint action** of A on $\mathrm{Lie}(G)$.*

Proof. Given $t \in \mathbb{R}$, We have that

$$e^{tAXA^{-1}} = Ae^{tX}A^{-1}$$

but $X \in \mathrm{Lie}(G)$ so $e^{tX} \in G$ whence $AXA^{-1} \in \mathrm{Lie}(G)$. □

Theorem 3.9. *Let G be a matrix Lie group. Then there exists a neighbourhood U of zero in $\mathrm{Lie}(G)$ and a neighbourhood V of $\mathbb{1}_n$ in G such that $\exp$ maps U homeomorphically onto V.*

Proof. Omitted. □

Corollary 3.10. *Let G be a path-connected matrix Lie group. Then every $A \in G$ admits a decomposition*

$$A = e^{X_1} \cdots e^{X_n}$$

for some $X_i \in \mathrm{Lie}(G)$.

Proof. Let E be the collection of all matrices $A \in G$ that admit a decomposition into the hypothesised form. By Theorem 3.9, E contains a neighbourhood of $\mathbb{1}_n$ say V. In particular, E is non-empty. Now, given any other $A \in E$, we have that AV is a neighbourhood of A and so E is open in G.

On the other hand, we claim that G is also closed. Suppose $\{ A_n \} \subseteq E$ is a sequence such that $A_n \to A \in G$ as $n \to \infty$. Clearly, A_n^{-1} for all n. Hence $A_n^{-1}A \to \mathbb{1}_n$ as $n \to \infty$. Hence, $A_n^{-1}A \in V$ for large enough n. Hence $A = A_n e^{X_n}$ for some $X_n \in \mathrm{Lie}(G)$. This is a product of elements in E so $A \in E$. Hence E contains all its limit points whence E is closed.

Finally, recall that the only clopen sets in the path-connected space X, the only clopen sets are $\varnothing$ and X. Since E is necessarily non-empty, we must have that $E = G$. Hence every element of G admits such a decomposition. □

Definition 3.11. Let G be a matrix Lie group. A **one-parameter subgroup** of G is a continuous group homomorphism $\varphi : \mathbb{R} \to G$.

Proposition 3.12. *Let G be a matrix Lie group and φ a one-parameter subgroup of G. Then there exists $X \in \mathrm{Lie}(G)$ such that $\varphi(t) = \exp(tX)$ for all $t \in \mathbb{R}$.*

Proof. By the definition of φ, we have that $\varphi(0) = \mathbb{1}_n$. Hence for sufficiently small t_0, we have $\varphi(t_0) \subseteq \exp(\mathrm{Lie}(G))$ by Theorem 3.9. Now let

$$X(t) = \frac{1}{t}\log(\varphi(t))$$

Then for $t \leq t_0$, we have $e^{tX(t)} \in \exp(\mathrm{Lie}(G))$. Now, given arbitrary $t \in \mathbb{R}$ and for all $N \geq 1$ we have

$$\varphi(t) = \varphi\left(\sum_{i=1}^{N} \frac{t}{N}\right) = \left[\varphi\left(\frac{t}{N}\right)\right]^N = \left[\exp\left(\frac{t}{N}X\left(\frac{t}{N}\right)\right)\right]^N = \exp\left(tX\left(\frac{t}{N}\right)\right)$$

We can now choose N so that $|t/N| \leq t_0$ so that the right hand side of the above is an element of $\exp^{\mathrm{Lie}(G)}$. $\square$

Corollary 3.13. *Let G be a matrix Lie group and φ a one-parameter subgroup of G. Then φ is smooth.*

Proposition 3.14. *Let G be a path-connected matrix Lie group and H an arbitrary matrix Lie group. If $\varphi : G \to H$ is a continuous homomorphism of matrix Lie groups then φ is smooth.*

Proof. First assume that $A \in G$ is contained in a neighbourhood of the identity $\mathbb{1}_n$. By Corollary 3.10, A admits a decomposition

$$A = e^{t_1 X_1} \cdots e^{t_n X_n}$$

where $t_i \in \mathbb{R}$ and $X_i \in \mathrm{Lie}(G)$. Then

$$\varphi(A) = \varphi(e^{t_1 X_1}) \cdots \varphi(e^{t_n X_n})$$

each of these is a one-parameter subgroup in the variables t_i. Since one-parameter subgroups are smooth and products of smooth functions are smooth, it follows that φ is smooth.

To see that φ is smooth on all of G, it suffices to realise that left-translation is a diffeomorphism of G so, any open neighbourhood B is diffeomorphic to an open neighbourhood of $\mathbb{1}_n$. $\square$

Theorem 3.15. *Let G_1 and G_2 be matrix Lie groups and $\varphi : G_1 \to G_2$ a homomorphism of matrix Lie groups. Then there exists a unique $\mathbb{R}$-linear map $\overline{\varphi} : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ such that*

1. $\varphi(e^X) = e^{\overline{\varphi}(X)}$ *for all* $X \in \mathrm{Lie}(G_1)$
2. $\overline{\varphi}(AXA^{-1}) = \varphi(A)\overline{\varphi}(X)\varphi(A)^{-1}$ *for all* $X \in \mathrm{Lie}(G_1), A \in G$
3. $\overline{\varphi}([X,Y]) = [\overline{\varphi}(X), \overline{\varphi}(Y)]$ *for all* $X, Y \in \mathrm{Lie}(G_1)$
4. $\overline{\varphi}(X) = \frac{d}{dt}(\varphi(e^{tX}))|_{t=0}$

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G_1)$ and $\mathfrak{h} = \mathrm{Lie}(G_2)$. Since φ is continuous, $\varphi(e^{tX})$ is a one-parameter of subgroup of G_2 for each $X \in G_1$. By Proposition 3.12, there exists a unique Z such that

$$\varphi(e^{tX}) = e^{tZ}$$

for all $t \in \mathbb{R}$. Since $\varphi(e^{tX}) \in G_2$, it follows that $Z \in \mathfrak{h}$. We shall take $\overline{\varphi}(X) = Z$.

Part 1: This follows directly from the definition of $\overline{\varphi}$ by taking $t = 1$. We shall use this to show that $\overline{\varphi}$ is a linear map. Let $s \in \mathbb{R}$. We need to show that $\overline{\varphi}(sX) = s\overline{\varphi}(X)$. We have that

$$e^{ts\overline{\varphi}(X)} = \varphi(e^{tsX}) = e^{t\overline{\varphi}(sX)}$$

for all $t \in \mathbb{R}$. We must next show that if $X, Y \in \mathfrak{g}$ then $\overline{\varphi}(X+Y) = \overline{\varphi}(X) + \overline{\varphi}(Y)$. For all $t \in \mathbb{R}$ we have

$$e^{t\overline{\varphi}(X+Y)} = e^{\overline{\varphi}(t(X+Y))} = \varphi(e^{t(X+Y)})$$

By the Trotter Product Formula and the fact that φ is continuous, we have

$$\begin{aligned} e^{t\overline{\varphi}(X+Y)} &= \varphi\left[\lim_{n\to\infty}\left(\exp\left(\frac{tX}{n}\right)\exp\left(\frac{tY}{n}\right)\right)^n\right] \\ &= \lim_{n\to\infty}\left[\varphi\left(\exp\left(\frac{tX}{n}\right)\right)\varphi\left(\exp\left(\frac{tY}{n}\right)\right)\right]^n \\ &= \lim_{n\to\infty}(e^{t\overline{\varphi}(X)/n}e^{t\overline{\varphi}(X)/n})^n \\ &= e^{t(\overline{\varphi}(X)+\overline{\varphi}(Y))} \end{aligned}$$

and so $\overline{\varphi}(X+Y) = \overline{\varphi}(X) + \overline{\varphi}(Y)$ as desired.

Part 2: We now show that $\overline{\varphi}(AXA^{-1}) = \varphi(A)\overline{\varphi}(X)\varphi(A)^{-1}$ for all $X \in \mathfrak{g}$ and $A \in G$. We have that

$$\begin{aligned} e^{t\overline{\varphi}(AXA^{-1})} &= e^{\overline{\varphi}(tAXA^{-1})} \\ &= \varphi(e^{tAXA^{-1}}) \\ &= \varphi(Ae^{tX}A^{-1}) \\ &= \varphi(A)\varphi(e^{tX})\varphi(A)^{-1} \\ &= \varphi(A)e^{t\overline{\varphi}(X)}\varphi(A)^{-1} \end{aligned}$$

and so $\overline{\varphi}(AXA^{-1}) = \varphi(A)\overline{\varphi}(X)\varphi(A)^{-1}$ as desired.

Part 3: We now show that $\overline{\varphi}([X,Y]) = [\overline{\varphi}(X), \overline{\varphi}(Y)]$ for all $X, Y \in \mathfrak{g}$. It is easy to see using the product rule that the following identity holds:

$$[X,Y] = \frac{d}{dt}(e^{tX}Ye^{-tX})\bigg|_{t=0}$$

Hence

$$\overline{\varphi}([X,Y]) = \overline{\varphi}\left(\frac{d}{dt}(e^{tX}Ye^{-tX})\bigg|_{t=0}\right) = \frac{d}{dt}\overline{\varphi}(e^{tX}Ye^{-tX})\bigg|_{t=0}$$

By Part 2 we then have

$$\begin{aligned}\overline{\varphi}([X,Y]) &= \frac{d}{dt}\varphi(e^{tX})\overline{\varphi}(Y)\varphi(e^{tX})\Big|_{t=0} \\ &= \frac{d}{dt}e^{t\overline{\varphi}(X)}\overline{\varphi}(Y)e^{-t\overline{\varphi}(X)}\Big|_{t=0} \\ &= [\overline{\varphi}(X),\overline{\varphi}(Y)]\end{aligned}$$

as required.

Part 4: This follows directly from the definition of $\overline{\varphi}$.

□

Theorem 3.16. *Let G_1 and G_2 be matrix Lie groups and $\varphi, \psi : G_1 \to G_2$ a homomorphism. Let $\overline{\varphi}, \overline{\psi} : \text{Lie}(G_1) \to \text{Lie}(G_2)$ be the induced linear maps of Lie algebras. If G_1 is path-connected and $\overline{\varphi} = \overline{\psi}$ then $\varphi = \psi$.*

Proof. Since G_1 is connected, every $A \in G_1$ admits a decomposition

$$A = e^{X_1} \dots e^{X_n}$$

where $X_i \in \text{Lie}(G_1)$. Now, if $\overline{\varphi} = \overline{\psi}$ we have

$$\varphi(e^{X_1} \cdots e^{X_n}) = e^{\overline{\varphi}(X_1)\cdots\overline{\varphi}(X_n)} = e^{\overline{\psi}(X_1)\cdots\overline{\psi}(X_n)} = \psi(e^{X_1} \cdots e^{X_n})$$

□

Theorem 3.17. *Let G_1 and G_2 be matrix Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. If G_1 is path-connected and simply connected then there exists a unique Lie group homomorphism $\phi : G_1 \to G_2$ such that $\overline{\phi} = \alpha$.*

Proof. Omitted. □

Corollary 3.18. *Let G_1 and G_2 be path-connected and simply connected matrix Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Then $G_1 \cong G_2 \iff \mathfrak{g}_1 \cong \mathfrak{g}_2$.*

Theorem 3.19. *Let G be a path-connected and simply connected matrix Lie group and $\mathfrak{g}$ its associated Lie algebra. If G' is a matrix Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ then $G' \cong G/Z$ for some discrete subgroup $Z \subseteq Z(G)$.*[1]

Proof. Omitted. □

Theorem 3.20. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of some matrix Lie group.*

[1] $Z(G)$ is the centre of G: all elements in G that commute

4 Abstract Lie Algebras

Throughout this section, we will let the base field of the Lie algebras be $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 4.1. Let $(L, [\cdot,\cdot])$ be a Lie algebra. We say that L is **abelian** if for all $x, y \in L$ we have $[x, y] = 0$.

Example 4.2. Let $(L, [\cdot,\cdot])$ be a one-dimensional Lie algbera. Then L is abelian. Indeed, Let x be the single basis element for L. Then, by anti-symmetry we have $[x, x] = -[x, x]$ so $[x, x] = 0$. Extending this by bilinearity, we see that $[x, y] = 0$ for all $x, y \in L$. Hence, over any field $\mathbb{F}$, there exists only one

Definition 4.3. Let $(L_1, [\cdot,\cdot]_1$ and $(L_2, [\cdot,\cdot]_2)$ be Lie algebras over $\mathbb{F}$. A **homomorphism** of Lie algebras from L_1 to L_2 is a $\mathbb{F}$-linear map $\varphi : L_1 \to L_2$ such that

$$[\varphi(x), \varphi(y)]_2 = \varphi([x, y]_1)$$

for all $x, y \in L_1$. We say that φ is an **isomorphism** if it is invertible and an **automorphism** if it is an isomorphism and $L_1 = L_2$.

Example 4.4. Let V be an $\mathbb{F}$-vector space. Then $\mathfrak{gl}(V) = \mathrm{End}(V)$ is a Lie algebra over $\mathbb{F}$ with Lie bracket given by commutator of endomorphisms. For example, if $V = \mathbb{F}^n$ then $\mathfrak{gl}(V) = \mathrm{Mat}(n, \mathbb{F})$.

Definition 4.5. Let L be a Lie algebra over $\mathbb{F}$. A **representation** of L on an $\mathbb{F}$-vector space V is a Lie algebra homomorphism $\varphi : L \to \mathfrak{gl}(V)$.

Definition 4.6. Let $(L, [\cdot,\cdot])$ be a Lie algebra. Let $U, V \subseteq L$ be linear subspaces of L. We define $[U, V]$ to be the linear subspace of L given by

$$[U, V] = \mathrm{span}_{\mathbb{F}} \{ [u, v] \mid u \in U, v \in V \}$$

Definition 4.7. Let $(L, [\cdot,\cdot])$ be a Lie algebra over $\mathbb{F}$. A subspace $H \subseteq L$ is a **sub-Lie algebra** if H is closed under the Lie bracket. In other words, $[H, H] \subseteq H$.

Definition 4.8. Let $(L, [\cdot,\cdot])$ be a Lie algebra over $\mathbb{F}$. A subspace $H \subseteq L$ is an **ideal** if $[L, I] \subseteq I$.

Remark. Note that an ideal of a Lie algebra is a sub-Lie algebra.

Lemma 4.9. *Let $(L_1, [\cdot,\cdot]_1)$ and $(L_2, [\cdot,\cdot]_2)$ be Lie algebras over $\mathbb{F}$. Then*

1. *If $I \subseteq L_1$ is an ideal then*

$$L_1/I = \{ x + I \mid x \in L_1 \}$$

   is a Lie algebra over $\mathbb{F}$ with Lie bracket given by $[x+I, y+I] = [x, y]_1 + I$. Furthermore. $\pi : L \to L_1/I$ is a surjective homomorphism of Lie algebras.

2. *If $\varphi : L_1 \to L_2$ is a homomorphism of Lie algebras then $\ker\phi$ is an ideal of L_1 and $\mathrm{im}\,\phi$ is a sub-Lie algebra of L_2. Furthermore,*

$$L_1/\ker\phi \cong \mathrm{im}\,\phi$$

3. *If $I, J \subseteq L_1$ are ideals then $I + J, I \cap J$ and $[I, J]$ are all ideals of L.*

4. *If $I, J \subseteq L_1$ are ideals then*

$$(I+J)/J \cong I/(I \cap J)$$

Proof.

Part 1: The quotient space is always an $\mathbb{F}$-vector space so we just need to show that the Lie bracket is bilinear, antisymmetric and satisfies the Jacobi identity. To this end, let $\lambda \in \mathbb{F}$ and $x, y, z \in L_1$. Then

$$\begin{aligned}[(x+I)+\lambda(y+I), z+I] = [(x+\lambda y)+I, z+I] &= [x+\lambda y, z]_1 + I \\ &= ([x,z]_1 + \lambda[y,z]_1) + I \\ &= ([x,z]_1 + I) + (\lambda[y,z]_1 + I) \\ &= [x,z] + \lambda[y,z]\end{aligned}$$

Hence $[\cdot,\cdot]$ is linear in the first argument. The same proof works for the second argument and so the Lie bracket is bilinear. For antisymmetricity, we have

$$[x+I, y+I] = [x,y]_1 + I = -[y,x]_1 + I = -([y,x]_1 + I) = -[y,x]$$

And finally, for the Jacobi identity, we have

$$\begin{aligned}&[x+I,[y+I,z+I]] + [z+I,[x+I,y+I]] + [y+I,[z+I,x+I]] \\ &= [x+I,([y,z]_1+I)] + [z+I,([x,y]+1+I)] + [y+I,([z,x]+I)] \\ &= ([x,[y,z]_1]_1 + I) + ([z,[x,y]_1]_1 + I) + ([y,[z,x]_1]_1 + I) \\ &= [x,[y,z]_1]_1 + [z,[x,y]_1]_1 + [y,[z,x]_1]_1 + I \\ &= 0 + I\end{aligned}$$

and so $[\cdot,\cdot]$ satisfies the Jacobi identity. We thus see that L_1/I is a Lie algebra.

Part 2: We first show that $\ker\varphi$ is an ideal of L_1. Let $x \in L_1$ and $y \in \ker\varphi$. We need to show that $[x,y] \in \ker\varphi$. We have that

$$\varphi([x,y]_1) = [\varphi(x), \varphi(y)]_2 = [\varphi(x), 0]_2 = 0$$

Now suppose that $x, y \in \operatorname{im}\varphi$. We need to show that $[x,y]_2 \in \operatorname{im}\varphi$. There exists $a, b \in L_1$ such that $\varphi(a) = x, \varphi(b) = y$ and so

$$[x,y]_2 = [\varphi(a), \varphi(b)]_2 = \varphi([a,b]_1) \in \operatorname{im}\varphi$$

We now tackle the isomorphism. Define

$$\begin{aligned}\overline{\pi} : L_1/\ker\pi &\to \operatorname{im}\pi \\ x + \ker\pi &\mapsto \pi(x)\end{aligned}$$

We claim that $\overline{\pi}$ is an isomorphism of Lie algebras. We must first check that the mapping is well-defined. In other words, if $x + \ker\pi = y + \ker\pi$ then $\overline{\pi}(x+\ker\pi) = \overline{\pi}(y+\ker\pi)$. By hypothesis, we have that $(y-x) + \ker\pi = 0 + \ker\pi$. We have that

$$\overline{\pi}(x+\ker\pi) = \pi(x) + \pi(y-x) = \pi(y) = \overline{\pi}(y+\ker\pi)$$

We must next check that $\overline{\pi}$ is indeed a homomorphism of Lie algebras. In other words, we must check that $\overline{\pi}$ preserves the Lie bracket structure. We have that

$$[\overline{\pi}(x+\ker\pi),\overline{\pi}(y+\ker\pi)]_2 = [\pi(x),\pi(y)]_2 = \pi([x,y]_1) = \overline{\pi}([x,y]_1+\ker\pi) = \overline{\pi}([x+\ker\pi,y+\ker\pi])$$

To show that $\overline{\pi}$ is an isomorphism, we need to show that it is both injective and surjective. For injectivity, it suffices to show that $\ker\overline{\pi}$ is trivial. To this end, let $x+\ker\pi \in \ker\overline{\pi}$. Then

$$0 = \overline{\pi}(x+\ker\pi) = \pi(x)$$

It follows that $x \in \ker\pi$ and so $x+\ker\pi = 0+\ker\pi$ whence $\ker\overline{\pi}$ is trivial. Now, for surjectivity, given $y \in \operatorname{im}\pi$, we need to find $x \in L_1/\ker\pi$ such that $\overline{\pi}(x) = y$. Since $y \in \operatorname{im}\pi$, there exists $x \in L_1$ such that $\pi(x) = y$. Then

$$\overline{\pi}(x+\ker\pi) = \pi(x) = y$$

and so $\overline{\pi}$ is surjective. We have thus shown that $\overline{\pi} : L_1/\ker\pi \to \operatorname{im}\pi$ is an isomorphism of Lie algebras.

Part 3: Let $a \in I, b \in J$ and $x \in L_1$. We need to show that $[x,a+b]_1 \in I+J$. We have that

$$[x,a+b]_1 = [x,a]_1 + [x,b] + 1 \in I+J$$

Next, let $a \in I \cap J$ and $x \in L_1$. We need to show that $[x,a] \in I \cap J$. Since I and J are ideals, $[x,a] \in I$ and $[x,a] \in J$ so $[x,a] \in I \cap J$.

Finally, let $a \in I, b \in J$ and $x \in L_1$. We need to show that $[x,[a,b]] \in [I,J]$. By the Jacobi identity, we have

$$[x,[a,b]] = -[b,[x,a]] - [a,[b,x]]$$

Now since I and J are ideals, $[x,a] \in I$ and $[b,x] \in J$. It then follows that the two terms in the right hand side of the above are elements of $[I,J]$.

Part 4: Let $i_1+j_1, i_2+j_2 \in I+J$. Considering these modulo J, we see that the j_i are irrelevant so we are interested in seeing how i_1 and i_2 can be equivalent modulo J. This happens exactly when $i_1-i_2 \in J$. But their difference is also an element of I so $i_1-i_2 \in I \cap J$.

□

Proposition 4.10. *Let $(L,[\cdot,\cdot])$ be a Lie algebra. Then the* ***center*** $Z(L) = \{\, x \in L \mid [x,y] = 0 \forall y \in L \,\}$ *and the* ***commutant*** $[L,L] = \{\, [x,y] \mid x,y \in L \,\}$ *are both ideals. In particular, $L/[L,L]$ is an abelian Lie algebra.*

Proof. Let $x \in L$ and $y \in Z(L)$. We need to show that $[x,y] \in Z(L)$. By definition, $[x,y] = 0$. Hence for all $z \in L$ we have $[z,[x,y]] = [z,0] = 0$ whence $[x,y] \in Z(G)$.

Now let $[x,y] \in [L,L]$ and $z \in L$. We need to show that $[z,[x,y]] \in [L,L]$. $[x,y]$ is clearly in L whence $[z,[x,y]] \in [L,L]$. It is easy to see that $L/[L,L]$ is abelian. Indeed, let $[\cdot,\cdot]'$ be the bracket of the quotient space $L/[L,L]$. We have for all $x+[L,L], y+[L,L] \in L/[L,L]$

$$[x+[L,L], y+[L,L]]' = [x,y] + [L,L] = 0 + [L,L]$$

and so $L/[L,L]$ is abelian. □

Example 4.11. Let $L = \mathfrak{gl}(n,\mathbb{F}) = \operatorname{Mat}(n,\mathbb{F})$. Then the Lie bracket is simply the matrix commutator. To find the commutant $[L,L]$, let $X,Y \in L$. Then $[X,Y] = XY - YX$. Note that $\operatorname{Tr} XY - YX = 0$ and so $[L,L] = \mathfrak{sl}(n,\mathbb{F})$. Now, since traceless matrices are always commutators of traceless matrices, it follows that $[\mathfrak{sl}(n,\mathbb{F}),\mathfrak{sl}(n,\mathbb{F})] = \mathfrak{sl}(n,\mathbb{F})$.

5 Solvable and Nilpotent Lie Algebras

Definition 5.1. Let L be a Lie algebra. We say that L is **simple** if L is non-abelian and has no non-trivial ideals. In other words, its only ideals are $\{ 0 \}$ and L.

Remark. If L is a simple Lie algebra then $Z(L) = 0$ and $[L, L] = L$. Indeed, the only ideals of L are $\{ 0 \}$ and L. Since L is non-abelian, $[L, L]$ is non-empty so we must have $[L, L] = L$. The only possibility for $Z(L)$, then, is $\{ 0 \}$.

Definition 5.2. Let $(L, [\cdot, \cdot])$ be a Lie algebra. We define the **derived series** of L, denoted D^iL to be the sequence of Lie algebras given by $D^0L = L, D^{i+1}L = [D^iL, D^iL]$ for $i \geq 0$. We say that L is **solvable** if $D^nL = \{ 0 \}$ for large enough n.

Proposition 5.3. *Let L be a Lie algebra. Then*

1. *D^iL is an ideal of L and in $D^{i-1}L$. Furthermore, $D^iL/D^{i+1}L$ is abelian.*
2. *If L is abelian then L is solvable.*
3. *If L is solvable then there exists a finite sequence of ideals $\{ 0 \} = I^n \subseteq I^{n-1} \cdots \subseteq I^1 \subseteq I^0 = L$ such that I^i/I^{i+1} is Abelian. In particular, $I^i = D^iL$.*

Proof.

Part 1: We first show that D^iL is an ideal in L. We shall prove this by induction on i. First suppose that $i = 0$. Then $D^0L = L$ which is clearly an ideal. Now suppose that it holds for $n - 1$. By the Jacobi identity[2], we have

$$\begin{aligned}[L, D^iL] = [L, [D^{i-1}L, D^{i-1}L]] &= -[D^{i-1}L, D^{i-1}L] - [D^{i-1}L, [L, D^{i-1}L]] \\ &\subseteq -D^iL - [D^{i-1}L, D^{i-1}L] \\ &\subseteq D^iL\end{aligned}$$

where we have used the induction hypothesis to see that $[L, D^{i-1}L] \subseteq D^{i-1}L$. Now we look at D^iL as an ideal in $D^{i-1}L$:

$$[D^{i-1}L, D^iL] = [D^{i-1}L, [D^{i-1}L, D^{i-1}L]] \subseteq [D^{i-1}L, D^{i-1}L] = D^{i-1}L$$

Where we have used the fact that $D^{i-1}L$ is an ideal in itself so that $[D^{i-1}L, D^{i-1}L] \subseteq D^{i-1}L$.

Finally, we show that $D^iL/D^{i+1}L$ is abelian. Let $[\cdot, \cdot]'$ be the Lie bracket on the quotient space $D^iL/D^{i+1}L$. Suppose that $x + D^{i+1}L, y + D^{i+1}L \in D^iL/D^{i+1}L$. Then

$$[x + D^{i+1}L, y + D^{i+1}L]' = [x, y] + D^{i+1}L$$

Now, $x, y \in D^iL$ and so $[x, y] \in [D^iL, D^iL] = D^{i+1}L$ whence $[x, y] + D^{i+1}L = 0 + D^{i+1}L$. It then follows that this quotient space is abelian.

Part 2: If L is abelian then $[L, L] = \{ 0 \}$ and so it solvable.

Part 3: This is just the first part restated.

□

[2] here we are abusing notation slightly, interpret addition and multiplication by scalar of subspaces by what they intuitively should be

Definition 5.4. Let $(L, [\cdot, \cdot])$ be a Lie algebra. We define the **lower central series** $D_i L$ of L to be the sequence of Lie algebras given by $D_0 L = L$ and $D_{i+1} L = [L, D_i L]$ for $i \geq 0$. We say that L is nilpotent if $D_n L = \{ 0 \}$ for sufficiently large n.

Proposition 5.5. *Let L be a Lie algebra. Then*

1. *$D_i L$ is an ideal of L and of $D_{i-1} L$.*

2. *If L is abelian then L is nilpotent.*

Proof. Part 1: We first show that $D_i L$ is an ideal in L. We shall prove this by induction on i. First suppose that $i = 0$. Then $D_0 L = L$ which is clearly an ideal in L. Now suppose that it holds true for $i - 1$. We have

$$[L, D_i L] = [L, [L, D_{i-1} L]] \subseteq [L, D_{i-1} L] = D_i L$$

where we have used the induction hypothesis to see that $[L, D_{i-1} L] \subseteq D_{i-1} L$.

To show that $D_i L$ is an ideal of $D_{i-1} L$, we have

$$[D_{i-1} L, D_i L] = [D_{i-1} L, [L, D_{i-1} L]] \subseteq [D_{i-1} L, L] = [L, D_{i-1} L] = D_i L$$

Part 2: Suppose that L is abelian. Then $D_0 L = [L, L] = \{ 0 \}$ and so L is nilpotent.

□

Proposition 5.6. *Let $(L, [\cdot, \cdot])$ be a Lie algebra. Then*

1. *If L is nilpotent then L is solvable.*

2. *If L is solvable (nilpotent) then all sub-Lie algebras, quotients L/I for some ideal $I \subseteq L$ and homomorphic images $\varphi(L)$ are solvable (nilpotent).*

3. *Let I be an ideal of L such that I is solvable and L/I is solvable. Then L is solvable.*

Proof. Part 1: It suffices to show that $D^i L \subseteq D_i L$. We prove this by induction. Let $i = 0$. Then clearly, $D^0 L = L = D_0 L$. Now, suppose that it holds for $i - 1$. We have that

$$D^i L = [D^{i-1} L, D^{i-1} L] \subseteq [L, D_{i-1} L] = D_i L$$

Now, since L is nilpotent, for large enough n, we have that

$$D^n L = D_i L = \{ 0 \}$$

whence L is solvable.

Part 2: We shall prove the statements about solvability, the same arguments follow through for nilpotency. Assume that L is solvable and $H \subseteq L$ is a sub-Lie algebra. We shall show by induction that $D^i H \subseteq D^i L$. If $i = 0$ then $D^0 H = H \subseteq L = D^0 L$. Now suppose it holds true for $i - 1$. We have that

$$D^i H = [D^{i-1} H, D^{i-1} H] \subseteq [D^{i-1} L, D^{i-1} L] = D^i L$$

Hence if $D^i H$ is solvable.

Now suppose that I is an ideal and let $[\cdot, \cdot]'$ be the commutator of the quotient space L/I. Recall that the natural quotient map $\pi : L \to L/I$ is surjective. It suffices to show that D^iL surjects onto D^iL/I. We shall prove this by induction. If $i = 0$ then $\pi(D^0L) = \pi(L) = L/I = D^0(L/I)$. Now suppose that it holds true for $i-1$. Then

$$\pi(D^iL) = \pi([D^{i-1}L, D^{i-1}L]) = [\pi(D^{i-1}L), \pi(D^{i-1}L)]' = [D^{i-1}(L/I), D^{i-1}(L/I)] = D^i(L/I)$$

Hence for large enough n, $D^n(L/I) = \{ 0 \}$.

The same idea works for any other surjective homomorphism of Lie algebras.

Part 3: Let $\pi : L \to L/I$ be the canonical surjection. Note that $\ker \pi = I$. Since L/I is solvable, we have that

$$\{ 0 \} = D^n(L/I) = \pi(D^nL)$$

for large enough n and so $D^nL \subseteq \ker \pi = I$. Now if I is solvable then $D^mI = \{ 0 \}$ for large enough m. Then

$$D^{n+m}(L) = D^m(D^n(L)) \subseteq D^m(I) = \{ 0 \}$$

and so L is solvable.

□

Example 5.7. Consider the following sets of matrices:

$$t(n, \mathbb{F}) = \{ A \in \mathrm{Mat}(n, \mathbb{F}) \mid A_{ij} = 0 \, \forall \, i > j \}$$
$$n(n, \mathbb{F}) = \{ A \in \mathrm{Mat}(n, \mathbb{F}) \mid A_{ij} = 0 \, \forall i \geq j \}$$

in other words, $t(n, \mathbb{F})$ and $n(n, \mathbb{F})$ are the collections of all $n \times n$ strictly upper triangular and upper triangular matrices respectively. Both are Lie algebras and $[t, t] \subseteq n$. Now note that n is nilpotent. Indeed, mutliplying strictly upper triangular matrices together pushes the non-zero elements towards the upper right corner of the matrix. This implies that taking repeated commutators of strictly upper triangular matrices eventually gives zero and so n is nilpotent. This implies that t is solvable. Indeed, $t/[t, t]$ is abelian and thus solvable. Furthermore, [t,t] is solvable since it is a sub-Lie algebra of the nilpotent Lie algebra n. Hence t is solvable.

Lemma 5.8. *Let V be an n-dimensional $\mathbb{F}$-vector space, $L \subseteq \mathfrak{gl}(n, \mathbb{F})$ a Lie algebra and $I \subseteq L$ an ideal. If $\lambda : I \to \mathbb{F}$ is a linear functional and*

$$W_{I,\lambda} = \{ v \in V \mid Yv = \lambda(Y)v \, \forall Y \in I \}$$

Then $LW_{I,\lambda} \subseteq W_{I,\lambda}$

Proof. We need to show that $Xw \in W_{I,\lambda}$ for all $w \in W_{I,\lambda}, x \in L$. In other words, we need to show that $YXw = \lambda(Y)Xw$ for all $Y \in I, x \in L$ and $w \in W_{I,\lambda}$. We have that

$$YXw = XYw + [Y, X]w = X(\lambda(y)w) + \lambda([Y, X])w = \lambda(y)Xw + \lambda([Y, X])w \qquad (1)$$

It thus suffices to prove that $\lambda([Y, X]) = 0$ for all $Y \in I, X \in L$. Since V is finite dimensional, there exists a least natural number m such that the sequence $w, Xw, X^2w, \ldots, X^mw$ is linearly independent. Let U be the subspace spanned by these vectors. Fix $0 \neq w \in W_{I,\lambda}$ and $X \in L$. We first claim that for all $Y \in I$ we have

$$YX^iw = \lambda(Y)X^iw + \sum_{j<i} \alpha_j X^jw$$

for some $\alpha_j \in \mathbb{F}$. We shall show this by induction on i. For $i = 0$, this is clear by the definition of $W_{I,\lambda}$. If $i = 1$ then this holds by Equation 1. Now suppose that it holds true up to $i - 1$. Then

$$YX^iw = [Y, X]X^{i-1}w + XYX^{i-1}w$$

Now, $[Y, X] \in I$ and so, by the induction hypothesis, we have

$$YX^iw = \lambda(Y)X^iw + \mathcal{X}^{\rangle-\infty}\sqsupseteq$$

which proves the claim. Now, this formula implies that, with respect to the basis $w, Xw, \ldots, X^mw$, any $y \in I$ is represented by an upper triangular matrix, say ρ_y, with the $\lambda(Y)$ on the diagonal. Then

$$\operatorname{Tr} \rho_y = (m + 1)\lambda(Y)$$

for all $Y \in I$. Furthermore, $[Y, X] \in I$ and so

$$(m + 1)\lambda([Y, X]) = \operatorname{Tr} \rho_{[Y,X]} = \operatorname{Tr}([\rho_Y, \rho_X]) = 0$$

This implies that $\lambda([Y, X]) = 0$ as required. □

Theorem 5.9 (Lie's Theorem)**.** *Let L be a solvable Lie algebra over $\mathbb{F}$ and $\rho : L \to \operatorname{End}(V)$ a representation of L on a complex vector space V. Then there exists a non-zero $v \in V$ which is a common eigenvector of $\rho(X)$ for all $X \in L$.*

Proof. We prove the theorem by induction on $\dim_{\mathbb{F}} L$. First suppose that $\dim L = 1$. Then $\rho(L)$ is one-dimensional. Picking an element of $\rho(L)$, we can put it into Jordan Normal Form and then all other elements of $\rho(L)$ will have a common eigenvector with this element (since they are simply constant multiples of this distinguished element).

Now suppose the claim is true for all solvable Lie algebras $\overline{L}$ of dimension $\dim \overline{L} \leq n-1$. Let $n = \dim L$. Since L is solvable, we have that $[L, L] \subsetneq L$ so we can choose non-zero $X \in L\backslash[L, L]$. Then we can write $L = \mathbb{F}X \oplus H$ for some ideal $H \subseteq L$ such that $[L, L] \subseteq H$. Then $\dim H = n-1$ and H is solvable. By the induction hypothesis, there exists $0 \neq v_0 \in V$ and $\lambda \in H^*$ such that $\rho(Y)v_0 = \lambda_Y v_0$ for all $Y \in H$. Now let

$$W_{H,\lambda} = \{\, v \in V \mid \rho(Y)v = \lambda(Y)v \ \forall\, Y \in H \,\}$$

By Lemma 5.8, this collection is L-invariant. This implies that $\rho(X)$ is a linear map from $W_{H,\lambda}$ to $W_{H,\lambda}$. This map can be put into Jordan Normal Form and thus has an eigenvector $w_0 \in W_{H,\lambda}$. Hence $\rho(X)w_0 = \lambda_X w_0$ for some $\lambda_X \in \mathbb{C}$. We can therefore use this to extend $\lambda \in H^*$ to $\overline{\lambda} \in L^*$ by setting $\overline{\lambda}|_H = \lambda$ and $\overline{\lambda}|_{\mathbb{F}X} = \lambda_X$. This is exactly what it means for $\rho(X)$ to have a common eigenvector for all $X \in L$. □

Corollary 5.10. *Let L be a Lie algebra. Then*

1. *If L is solvable and $\rho : L \to \operatorname{End}(V)$ is a representation on a $\mathbb{C}$-vector space V of L then there exists a basis of V in which all $\rho(X)$ is upper triangular for all $X \in L$.*

2. *If L is solvable then any irreducible complex representation of L is one-dimensional.*

3. *L is solvable if and only if $[L, L]$ is nilpotent.*

Proof. Omitted. □

Definition 5.11. Let L be a Lie algebra and $\rho : L \to \text{End}(V)$ a representation of L on a vector space V. We say that a subspace $W \subseteq V$ is **invariant** if $\rho(X)W \subseteq W$ for all $X \in L$. Furthermore, we say that ρ is **irreducible** if $\{ 0 \}$ and V are the only invariant subspaces of V.

Theorem 5.12 (Engel's Theorem)**.** *Let L be a Lie algebra. Then L is nilpotent if and only if for all $X \in L$, $\text{ad}_X : L \to L$ is nilpotent.*

6 Semisimple Lie Algebras and Killing Forms

Definition 6.1. Let L be a Lie algebra over $\mathbb{F}$. We say that L is **semisimple** if it has no non-zero solvable ideals. Furthermore, we say that L is **reductive** if any solvable ideal is contained in the centre $Z(L)$.

Proposition 6.2. *Let L be a Lie algebra over $\mathbb{F}$. Then L has a unique maximal solvable ideal called the **radical ideal** and denoted* $\text{rad}(L)$*.*

Proof. Let I_1 and I_2 be solvable ideals in L. We first clam that $I_1 + I_2$ is solvable. Recall that we have an isomorphism of ideals

$$\frac{I + 1 + I_2}{I_1} \cong \frac{I_2}{I_1 \cap I_2}$$

The right hand side is clearly solvable as the quotient of a solvable Lie algebra. Since I_1 is solvable, it follows that $I_1 + I_2$ is solvable. By induction, we see that the sum of all solvable ideals is solvable and is clearly the maximal unique such ideal. □

Proposition 6.3. *Let L be a Lie algebra. Then*

1. *L is semisimple if and only if* $\text{rad}(L) = \{ 0 \}$.
2. *L is reductive if and only if* $\text{rad}(L) = Z(L)$.
3. *L simple $\implies$ L semisimple $\implies$ L reductive.*
4. $L/\text{rad}(L)$ *is semisimple.*
5. *L is semisimple if and only if it has no non-zero abelian ideals.*

Proof.

Part 1: First suppose that L is semisimple. Then L has no non-zero solvable ideals so, clearly, $\text{rad}(L) = \{ 0 \}$. Conversely, any solvable ideal is necessarily contained in $\text{rad}(L)$. But $\text{rad}(L) = \{ 0 \}$ so this is the only possible solvable ideal.

Part 2: First suppose that L is reductive. Then every solvable ideal of L is contained in $Z(L)$. In particular, $\text{rad}(L) \subseteq Z(L)$. Now $Z(L)$ is an ideal of L that is also an abelian sub-Lie algebra so $Z(L)$ is itself solvable. Hence $Z(L) \subseteq \text{rad}(L)$ and so $\text{rad}(L) = Z(L)$. Conversely, suppose that $\text{rad}(L) = Z(L)$. Then every solvable ideal of L is contained in $Z(L)$ whence L is reductive.

Part 3: If L is simple then it has no non-trivial ideals. Hence the only possible solvable ideal is $\{ 0 \}$ whence L is semisimple. Furthermore, $\{ 0 \} \subseteq Z(L)$ and so L is reductive.

Part 4: Let I be a solvable ideal of $L/\text{rad}(L)$. Let $\pi : L \to L/\text{rad}(L)$ be the canonical surjection. Consider the ideal $\overline{I} = \pi^{-1}(I)$. Then $\text{rad}(L) \subseteq \overline{I}$ and $\overline{I}/\text{rad}(L) = I$ whence $\overline{I}$ is solvable. But then $\text{rad}(L) = \overline{I}$ whence $I = \{ 0 \}$.

Part 5 First suppose that L is semisimple. Then the only solvable ideals of L is $\{ 0 \}$. Since any abelian ideal would necessarily be solvable, the only possible abelian ideal of L is $\{ 0 \}$.

Conversely, assume that L is not semisimple. Then there exists a non-zero solvable ideal I. In other words, $D^n I = \{0\}$ for sufficiently large n. Let N be smallest such N. Then $\{0\} = D^N I = [D^{N-1}I, D^{N-1}I]$ and so $D^{N-1}I$ is an abelian ideal in L. □

Example 6.4. $L = \mathfrak{sl}(n, \mathbb{F})$ is simple. We shall prove this in the case that $n = 2$. Recall that this Lie algebra consists of all traceless matrices in $\mathrm{Mat}(2, \mathbb{F})$. This has a basis given by matrices of the form

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy the relations $[H, E] = F$, $[F, H] = 2H$, $[F, E] = -2E$. Now let $I \subseteq L$ be a non-zero ideal and let $x = \alpha H + \beta E + \gamma F \in I$ for $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero. First suppose that both α and β are zero. Then γ is non-zero $z = \gamma F \in I$ whence $F \in I$. Now, $[E, F] = 2E \in I$ and so also $E \in I$. Furthermore, $[H, F] = -2H \in I$ and so $H \in I$. It then follows that $L \subseteq I$ whence $L = I$.

Now suppose that $c = 0$. Then

$$[H, \alpha H + \beta E] = \alpha[H, H] + \beta[H, E] = \beta F \in I \implies F \in I$$

It then follows that $[H, F] = -2H \in I$ and so $H \in I$. Similarly, $[E, F] = 2E$ so $E \in I$. Hence $L \subseteq I$ whence $L = I$ in this case also. We thus see that the only ideals of L are $\{0\}$ and L whence L is simple.

Example 6.5. $L = \mathfrak{gl}(n, \mathbb{F}) = \mathrm{Mat}(n, \mathbb{F})$ is reductive. Indeed, given any $A \in L$ we can decompose it into the form

$$A = \frac{1}{n}\mathrm{Tr}(A)\mathbb{1}_n + \left(A - \frac{1}{n}\mathrm{Tr}(A)\mathbb{1}_n\right)$$

the first term in this decomposition is clearly in $Z(L)$ since it is a multiple of the identity matrix. The second term is clearly in $\mathfrak{sl}(n, \mathbb{F})$. Since $\mathfrak{sl}(n, \mathbb{F})$ is simple, any proper solvable ideal of L must lie in $Z(L)$ whence L is reductive.

Definition 6.6. Let L be a Lie algebra and $\rho : L \to \mathrm{End}(V)$ a representation of L on an $\mathbb{F}$-vector space. We can define a bilinear form

$$\begin{aligned} B_\rho : L \times L &\to \mathbb{F} \\ (X, Y) &\mapsto \mathrm{Tr}(\rho(X) \circ \rho(Y)) \end{aligned}$$

Proposition 6.7. *Let L be a Lie algebra and ρ_V a representation of L. Then B_ρ is symmetrix and invariant:*

$$B_\rho([X, Y], Z) = B_\rho(X, [Y, Z])$$

for all $X, Y, Z \in L$.

Proof. Let $X, Y \in L$. Since the trace is invariant under cyclic permutation, we have

$$B_\rho(X, Y) = \mathrm{Tr}(\rho(X) \circ \rho(Y)) = \mathrm{Tr}(\rho(Y) \circ \rho(X)) = B_\rho(Y, X)$$

Now let $Z \in L$. We have

$$\begin{aligned}
B_\rho([X,Y],Z) &= \mathrm{Tr}(\rho([X,Y]) \circ \rho(Z)) \\
&= \mathrm{Tr}([\rho(X),\rho(Y)] \circ \rho(Z)) \\
&= \mathrm{Tr}((\rho(X)\circ\rho(Y) - \rho(Y)\circ\rho(X)) \circ \rho(Z)) \\
&= \mathrm{Tr}(\rho(X)\circ\rho(Y)\circ\rho(Z)) - \mathrm{Tr}(\rho(Y)\circ\rho(X)\circ\rho(Z)) \\
&= \mathrm{Tr}(\rho(X)\circ\rho(Y)\circ\rho(Z)) - \mathrm{Tr}(\rho(X)\circ\rho(Z)\circ\rho(Y)) \\
&= \mathrm{Tr}(\rho(X)\circ\rho(Y)\circ\rho(Z) - \rho(X)\circ\rho(Z)\circ\rho(Y)) \\
&= \mathrm{Tr}(\rho(X)\circ[\rho(Y),\rho(Z)]) \\
&= \mathrm{Tr}(\rho(X)\circ\rho([Y,Z])) \\
&= B_\rho(X,[Y,Z])
\end{aligned}$$

□

Proposition 6.8. *Let L be a Lie algebra. Then given $X \in L$, the adjoint action $\mathrm{ad}_X(Z) = [X,Z]$ is a representation of L on itself.*

Proof. We need to show that

$$\mathrm{ad} : L \to \mathrm{End}(L) X \mapsto \mathrm{ad}_X$$

is a homomorphism of Lie algebras. It is clearly a linear map by linearity of the Lie bracket. Note that the Lie bracket in $\mathrm{End}(L)$ is given by the commutator of endomorphisms. We need to show that $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. For all $Z \in L$, the Jacobi identity implies

$$\begin{aligned}
\mathrm{ad}_{[X,Y]}(Z) = [[X,Y],Z] = -[Z,[X,Y]] = [X,[Y,Z]] + [Y,[Z,X]] &= [X,[Y,Z]] - [Y,[X,Z]] \\
&= \mathrm{ad}_X \circ \mathrm{ad}_Y(Z) - \mathrm{ad}_Y \circ \mathrm{ad}_X(Z)
\end{aligned}$$

as required. □

Definition 6.9. Let L be a Lie algebra, $X \in L$ and $\mathrm{ad}_X : L \to \mathrm{End}(L)$ the adjoint representation of L on itself. We define the **Killing form** of L to be $\kappa_L = B_{\mathrm{ad}}$.

Proposition 6.10. *Let $(L_1, [\cdot,\cdot]_1)$ and $(L_2, [\cdot,\cdot]_2$ be Lie algebras and $\varphi : L_1 \to L_2$ an isomorphism of Lie algebras. Then*

$$\kappa_{L_2}(\varphi(X),\varphi(Y)) = \kappa_{L_1}(X,Y)$$

for all $X, Y \in L_1$.

Proof. Let $X, Y \in L_1$. We have that

$$\kappa_{L_2}(\varphi(X),\varphi(Y)) = \mathrm{Tr}(\mathrm{ad}_{\varphi(X)} \circ \mathrm{ad}_\varphi(Y))$$

Now let $Z \in L_1$ and let $Z' = \varphi(Z)$. Observe

$$\mathrm{ad}_{\varphi(X)}(Z') = [\varphi(X), Z']_2 = [\varphi(X),\varphi(Z)]_2 = \varphi([X,Z]_1) = \varphi \circ \mathrm{ad}_X(Z) = \varphi \circ \mathrm{ad}_X \circ \varphi^{-1}(Z')$$

We thus see that $\mathrm{ad}_{\varphi(X)} = \varphi \circ \mathrm{ad}_X \circ \varphi^{-1}$. It then follows that

$$\begin{aligned}
\kappa_{L_2}(\varphi(X),\varphi(Y)) &= \mathrm{Tr}(\varphi\circ\mathrm{ad}_X\circ\varphi^{-1}\circ\varphi\circ\mathrm{ad}_Y\circ\varphi^{-1}) \\
&= \mathrm{Tr}(\varphi\circ\mathrm{ad}_X\circ\mathrm{ad}_Y\circ\varphi^{-1}) \\
&= \mathrm{Tr}(\mathrm{ad}_X\circ\mathrm{ad}_Y) \\
&= \kappa_{L_1}(X,Y)
\end{aligned}$$

□

Lemma 6.11. *Let L be a Lie algebra and κ its Killing form. If $I \subseteq L$ is an ideal then*

$$I^{\perp} = \{ X \in L \mid \kappa(X, Y) = 0 \; \forall \, y \in I \}$$

is an ideal of L.

Proof. Let $Y \in I^{\perp}$. We need to show that $[X, Y] \in I^{\perp}$ for all $X \in L$. For all $Z \in I$, we have

$$\kappa([X, Y], Z) = -\kappa([Y, X], Z) = -\kappa(Y, [X, Z]) = 0$$

where we have used the invariance of κ and the fact that $[X, Z] \in I$. Thus $[X, Y] \in I^{\perp}$. □

Theorem 6.12 (Cartan's Criterion)**.** *Let L be a Lie algebra and κ its Killing form. Then*

1. *L is solvable if and only if $\kappa(X, Y) = 0$ for all $x \in L, Y \in [L, L]$.*

2. *L is semisimple if and only if κ is non-degenerate.*

Proof. Omitted. □

Corollary 6.13. *Let L be a Lie algebra. Then*

1. *If L is semisimple and $I \subseteq L$ is an ideal then $L = I \oplus I^{\perp}$.*

2. *L is semisimple if and only if L is a direct sum of simple Lie algebras. Such a direct summand decomposition is uniquely determined by L.*

3. *If L is semisimple then $L = [L, L]$.*

4. *If $L = \bigoplus_i L_i$ is semisimple then any ideal of L is a direct sum of some of the L_i.*

5. *If L is semisimple then any ideal, quotient and homomorphic image of L is also semisimple.*

Proof.

Part 1: Let $J = I \cap I^{\perp}$. Then $\kappa_L|_J = 0$ by the definition of $I^{\perp}$. It then follows that J is solvable by Cartan's Criterion. Since L is semisimple, $\mathrm{rad}(L) = \{ 0 \}$ and so $I \cap I^{\perp} = J = \{ 0 \}$. Furthermore, $\dim I^{\perp} = \dim L - \dim I$ so $L = I \oplus I^{\perp}$.

Part 2: First suppose that L is semisimple. Then we may apply Part 1 repeatedly until each direct summand has no non-trivial ideals. Conversely, suppose that L admits a direct summand decomposition $L = \bigoplus_i L_i$ into simple Lie algebras. Then the Killing form of L is given by the sum of the Killing forms of the L_i. Since any simple Lie algebra is semisimple, the Killing form of each L_i is non-degenerate whence the Killing form of L is non-degenerate. Applying Cartan's Criterion again, we see that L is semisimple.

Part 3: Since L is the direct sum of simple Lie algebras L_i and each L_i satisfies $L_i = [L_i, L_i]$, the same must be true of L by linearity of the Lie bracket.

Part 4: We shall prove this by induction on the number of direct summands in the decomposition of L into simple Lie algebras. Suppose L admits the decomposition $L = \bigoplus_{i=1}^{k} L_i$. First suppose that $k = 1$. Then L is simple and its only ideals are $\{ 0 \}$ and L which both appear in the direct sum decomposition of L (trivially). Now suppose the statement holds true for $k - 1$. Let $\pi_k : L \to L_k$ be the projection of L onto its k^{th} direct summand. Then $\pi_k(I) \subseteq L_k$ is an ideal. Since L_k is simple, either $\pi_K(I) = \{ 0 \}$ or $\pi_K(I) = L_k$. In the former case, it follows that $I \subseteq L_1 \oplus \cdots \oplus L_{k-1}$. Applying the induction hypothesis, we see that I is equal to a direct sum of some of these summands. In the second case we have that $[L_k, I] = [L_k, \pi_k(I)] = [L_k, L_k] = L_k$. Since I is an ideal, it follows that $L_k \subseteq I$. Hence $I = L_k \oplus I'$ for some $I' \in L_1 \oplus \cdots \oplus L_{k-1}$. The claim then follows by again invoking the induction hypothesis.

□

7 Complexification and Real Forms

Definition 7.1. Let V be an $\mathbb{R}$-vector space. We define the **complexification** of V, denoted $V_{\mathbb{C}}$ to be the $\mathbb{C}$-vector space given by $V_{\mathbb{C}} = V \times V$ with compontent wise addition and scalar multiplication

$$(a+ib)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$$

for $a, b \in \mathbb{R}$. We shall use the notation $(v_1, v_2) = v_1 + iv_2$.

Remark. If $\mathfrak{g}$ is an $\mathbb{R}$-Lie algebra then $\mathfrak{g}_{\mathbb{C}}$ is a complex Lie algebra. Indeed, all we have to do is extend $[\cdot, \cdot]$ to be $\mathbb{C}$-bilinear:

$$[v_1 + iv_2, w_1 + iw_2] = [v_1, w_1] - [v_2, w_2] + i[v_1, w_2] + i[v_2, w_1]$$

Furthermore, if $\varphi : \mathfrak{g} \to W$ is a linear map where W is a complex vecctor space then φ induces a mapping

$$\varphi : \mathfrak{g}_{\mathbb{C}} \to W$$

by setting $\varphi(v_1 + iv_2) = \varphi(v_1) + i\varphi(v_2)$. In particular, a complex representation of $\mathfrak{g}$ extends to a complex representation of $\mathfrak{g}_{\mathbb{C}}$. This then implies that the bilinear form B_ρ (and thus the Killing form) extend to $\mathfrak{g}_{\mathbb{C}}$.

Definition 7.2. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then $\mathfrak{g}$ is called a **real form** of $\mathfrak{g}_{\mathbb{C}}$.

Theorem 7.3. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is solvable (nilpotent, semisimple) if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable (nilpotent, semisimple).*

Proof. Solvability and nilpotency concern vanishing of nested and repeated commutators. This happens over $\mathbb{R}$ if and only if it happens over $\mathbb{C}$. For semisimplicity, note that $\kappa_{\mathfrak{g}}$ is non-degenerate if and only if $\kappa_{\mathfrak{g}_{\mathbb{C}}}$ is non-degenerate. We may thus appeal to Cartan's criterion to see that $\mathfrak{g}_{\mathbb{C}}$ is semisimple if and only if $\mathfrak{g}$ is semisimple $\square$

Example 7.4. Let $\mathfrak{g} = \mathrm{Lie}(\mathrm{U}(n)) = \{ X \in \mathrm{Mat}(n, \mathbb{C}) \mid X = -X^\dagger \}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. Indeed, if $A \in \mathfrak{g}_{\mathbb{C}}$ then we can write

$$A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$$

The second term on the right hand side is clearly an element of $\mathrm{Lie}(\mathrm{U}(n))$ whereas the first term is an element of $i\mathrm{Lie}(\mathrm{U}(n))$.

8 Representation Theory of Lie Algebras

Definition 8.1. Let $\mathfrak{g}$ be a Lie algebra and ρ_V a representation of $\mathfrak{g}$. We say that ρ_V is **faithful** if $\ker \rho = \{ 0 \}$.

Definition 8.2. Let $\mathfrak{g}$ be a Lie algebra and ρ_{V_1}, ρ_{V_2} representations of $\mathfrak{g}$. A linear map $f : V_1 \to V_2$ is said to be an **intertwiner** between ρ_{V_1} and ρ_{V_2} if

$$\rho_{V_2}(X) \circ f = f \circ \rho_{V_1}(X)$$

for all $X \in \mathfrak{g}$.

Definition 8.3. Let $\mathfrak{g}$ be a Lie algebra and ρ_{V_1}, ρ_{V_2} representations of $\mathfrak{g}$. We say that ρ_{V_1} and ρ_{V_2} are **equivalent** if there exists an invertible intertwiner between them.

Example 8.4. Let $\mathfrak{g}$ be a Lie algebra. Then we have the **trivial** representation $\rho : \mathfrak{g} \to \operatorname{End}(\mathbb{C})$ defined by $\rho(x) = 0$ for all$x \in \mathfrak{g}$.

Example 8.5. Let $\mathfrak{g} \subseteq \operatorname{Mat}(n, \mathbb{F})$ be a Lie algebra. Then we have the **fundamental** given by

$$\begin{aligned} \rho : \mathfrak{g} &\to \operatorname{End}(\mathbb{F}^n) \\ g &\mapsto g \end{aligned}$$

Definition 8.6. Let $\mathfrak{g}$ be a Lie algebra and ρ_{V_1},ρ_{V_2} be representations of $\mathfrak{g}$. We define the **direct sum** representation, denoted $\rho_{V_1} \oplus \rho_{V_2}$ be the representation of $\mathfrak{g}$ on $V_1 \oplus V_2$ given by $\rho(x)(v_1, v_2) = (\rho_{V_1}(x)v_1, \rho_{V_2}(x)v_2)$.

Lemma 8.7. *Let $\mathfrak{g}$ be a Lie algebra and ρ_{V_1}, ρ_{V_2} representations of $\mathfrak{g}$. If $f : V_1 \to V_2$ is an intertwiner between ρ_{V_1} and ρ_{V_2} then $\ker f$ and $\operatorname{im} f$ are invariant subspaces of V_1 and V_2 respectively.*

Proof. Let $x \in \mathfrak{g}$. $v_1 \in \ker f$ if and only if $f(v_1) = 0$. Then $\rho_{V_1}(x)v_1 \in \ker f$. Indeed, we have

$$f(\rho_{V_1}(x)v_1) = \rho_{V_2}(x)f(v_1) = 0$$

Furthermore, $v_2 \in \operatorname{im} f$ if and only if there exists $v_1 \in V_1$ such that $f(v_1) = v_2$. Then

$$\rho_{V_2}(x)v_2 = \rho_{V_2}(x)f(v_1) = f(\rho_{V_1}(x)v_1) \in \operatorname{im} f$$

Hence they are both invariant subspaces of their respective ambient spaces. □

Theorem 8.8 (Schur's Lemma)**.** *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If ρ_{V_1}, ρ_{V_2} are irreducible representations of $\mathfrak{g}$ and $f : V_1 \to V_2$ is an intertwiner between them then either $f \equiv 0$ or f is an isomorphism.*

2. *If ρ_V is an irreducible representation of $\mathfrak{g}$ on a $\mathbb{C}$-vector space V and $f : V \to V$ is a self-intertwiner of ρ_1 then $f = \lambda \mathbb{1}_V$ for some $\lambda \in \mathbb{C}$.*

3. *If ρ_{V_1}, ρ_{V_2} are complex representations of $\mathfrak{g}$ and $f : V_1 \to V_2, \overline{f} : V_1 \to V_2$ are non-zero intertwiners between them then $f = \lambda \overline{f}$ for some $\lambda \in \mathbb{C}$.*

Proof.

Part 1: By Lemma 8.7, $\ker f$ is an invariant subspace of V_1. Since ρ_{V_1} is irreducible, we must either have that $\ker f = \{0\}$ or $\ker f = V_1$. In the first case, $\ker f$ is trivial whence f is injective. In the latter case, $f \equiv 0$. Furthermore, $\operatorname{im} f$ is an invariant subspace of V_2. Since ρ_{V_2} is irreducible, either $\operatorname{im} f = \{0\}$ or $\operatorname{im} f = V_2$. In the first case, $f \equiv 0$ and in the second case, f is surjective. We thus have that either $f \equiv 0$ or f is surjective and injective whence it is an isomorphism.

Part 2: Since $f : V_1 \to V_1$ is a linear map of $\mathbb{C}$-vector spaces, it has at least one eigenvector $v_\lambda \in V_1$ with eigenvalue $\lambda \in \mathbb{C}$. By definition, we have that $f \circ \rho_V = \rho_V \circ f$. Since $\mathbb{1}_V$ commutes with any linear map, we have that

$$(f - \lambda \mathbb{1}_V) \circ \rho_V = \rho_V \circ (f - \lambda \mathbb{1}_V)$$

and so $f - \lambda \mathbb{1}_V$ is also a self-intertwiner of ρ_V. Since ρ_V is irreducible, Part 1 implies that $f - \lambda \mathbb{1}_V$ is either the zero map or an isomorphism. But $f - \lambda \mathbb{1}_V$ cannot be an isomorphism since $(f - \lambda \mathbb{1}_V)v_\lambda = 0$. Hence $f - \lambda \mathbb{1}_V \equiv 0$ and so $f = \lambda \mathbb{1}_V$.

Part 3: If f and $\overline{f}$ are two non-zero intertwiners between ρ_{V_1} and ρ_{V_2} then f and $\overline{f}$ are necessarily isomorphisms. This implies that f^{-1} exists. We claim that $f^{-1} \circ \overline{f}$ is a self-intertwiner of ρ_{V_1}. To this end, let $x \in \mathfrak{g}$. Note that f^{-1} is an intertwiner between ρ_{V_2} and ρ_{V_1}. Then

$$\rho_{V_1}(x) \circ f^{-1} \circ \overline{f} = f^{-1} \circ \rho_{V_2}(x) \circ \overline{f} = f^{-1} \circ \overline{f} \circ \rho_{V_1}$$

and so $f^{-1} \circ \overline{f}$ is a self-intertwiner of ρ_{V_1}. Part 2 then implies that $f^{-1} \circ \overline{f} = \mathbb{1}_V$ whence $f = \lambda \overline{f}$ for some $\lambda \in \mathbb{C}$.

□

9 Representations of $\mathfrak{sl}(2, \mathbb{C})$

In this section, we aim to classify all finite-dimensional complex irreducible representations of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Recall that

$$\mathfrak{sl}(2, \mathbb{C}) = \{ A \in \mathrm{Mat}(2, \mathbb{C}) \mid \mathrm{Tr}\, A = 0 \}$$

and has a $\mathbb{R}$-basis given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

These matrices satisfy the relations $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$. Let $\rho_V : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$. Given $x \in \mathfrak{g}$ and $v \in V$, we shall write $\rho_V(X)v = X \cdot v$ for short. Note that since V is a complex vector space, $\rho_V(H)$ necessarily has an eigenvector, say $v \neq 0 \in V$ with corresponding eigenvalue λ: $H \cdot v = \lambda v$.

Proposition 9.1. *With notation as above, E and F are* ***raising*** *and* ***lowering*** *operators respectively. In other words, $E^m \cdot v = 0$ or $E^m \cdot v$ is an eigenvector for $\rho_V(H)$ with eigenvalue $\lambda + 2m$. Similarly, $F^m \cdot v = 0$ or $F^m \cdot v$ is an eigenvector for $\rho_V(H)$ with eigenvalue $\lambda - 2m$.*

Proof. We first show that E is a raising operator. We shall prove this by induction on m. First suppose that $m = 1$. We have that

$$H \cdot E \cdot v = [H, E] \cdot v + E \cdot H \cdot v = 2E \cdot v + E \cdot (\lambda v) = 2E \cdot v + \lambda(E \cdot v) = (\lambda + 2)E \cdot v$$

Now suppose that the claim holds for $m - 1$. By the induction hypothesis, we have

$$\begin{aligned} H \cdot E^m \cdot v = H \cdot E \cdot E^{m-1} v &= ([H, E] + E \cdot H) \cdot E^{m-1} v \\ &= [H, E] \cdot E^{m-1} v + E \cdot H \cdot E^{m-1} v \\ &= 2E \cdot E^{m-1} v + E \cdot (\lambda + 2(m-1)) E^{m-1} v \\ &= 2E^m v + (\lambda + 2(m-1)) E^m v \\ &= (\lambda + 2m) E^m v \end{aligned}$$

The exact same argument follows through to show that F is a lowering operator. □

Proposition 9.2. *With notation as above, $\rho(H)$ can only have integer eigenvalues.*

Proof. Note that since eigenvectors for different eigenvalues must be linearly independent and V is finite-dimensional, we must have that $E^m v = 0$ for large enough m and $F^k v = 0$ for large enough k. Let m be the smallest natural number such that $E^m v = 0$ but $E^{m-1} v = v_0 \neq 0$. Consider the sequence of vectors $v_0, v_1 = Fv_0, \ldots, v_l = F^l v_0, \ldots$ and also set $v_{-1} = 0$. Let λ_0 be such that $H \cdot v_0 = \lambda_0 v_0$. We first claim that $H \cdot v_l = (\lambda_0 - 2l)v_l$. We prove this by induction on l. First suppose that $l = 0$. Then, by definition, $H \cdot v_0 = \lambda_0 v_0$ and this satisfies the claim. Now suppose the claim holds true for $l - 1$. We have

$$\begin{aligned} H \cdot v_l = H \cdot F^l \cdot v_0 = H \cdot F \cdot F^{l-1} \cdot v_0 &= ([H, F] + F \cdot H) \cdot F^{l-1} v_0 \\ &= [H, F] \cdot F^{l-1} \cdot v_0 + F \cdot H \cdot F^{l-1} v_0 \\ &= -2F \cdot F^{l-1} \cdot v_0 + F \cdot (\lambda_0 - 2(l-1))F^{l-1} v_0 \\ &= (\lambda_0 - 2l)F^l v_0 \\ &= (\lambda_0 - 2l)v_l \end{aligned}$$

as required. We next claim that $E \cdot v_l = l(\lambda_0 - l + 1)v_{l-1}$. We shall again prove this by induction on l. First suppose that $l = 0$. Then $E \cdot v_0 = E^m v_0 = 0$ and so the claim holds. Now suppose it holds for $l - 1$. We have that

$$\begin{aligned} E \cdot v_l = E \cdot F^l v_0 = E \cdot F \cdot v_{k-1} &= ([E, F] + F \cdot E) \cdot v_{l-1} \\ &= [E, F] \cdot v_{l-1} + F \cdot E \cdot v_{l-1} \\ &= H \cdot v_{l-1} + F \cdot (l-1)(\lambda_0 - l + 2)v_{l-1} \\ &= (\lambda_0 - 2l + 2)v_{l-1} + (l-1)(\lambda_0 - l + 2)v_{l-1} \\ &= (\lambda_0 - 2l + 2 + l\lambda_0 - \lambda_0 - l^2 + l + 2l - 2)v_{l-1} \\ &= (l\lambda_0 - l^2 + l)v_{l-1} \\ &= l(\lambda_0 - l + 1)v_{l-1} \end{aligned}$$

as required. Since V is finite-dimensional and v_l are all eigenvectors of H with distinct eigenvalues, $v_k = 0$ for large enough k. Assume that $v_{k-1} = 0$. Then

$$E \cdot v_k = k(\lambda_0 - k + 1)v_{k-1}$$

Observe that, since $v_{k-1} \neq 0$ then $E \cdot v_k \neq 0$ if $\lambda_0 \notin \mathbb{Z}$. But then this would imply that $v_k \neq 0$ which is a contradiction. Hence we must have that $\lambda_0 = 0$. □

Proposition 9.3. *Let $g = \mathfrak{sl}(2, \mathbb{C})$. For each $n = 1, 2, \ldots,$ we have a representation of $\mathfrak{g}$ on $V_n = \mathbb{C}^n$ given by*

$$\begin{aligned} \rho_n(H)e_m &= (n - 1 - 2m)e_m \\ \rho_n(E)e_m &= m(n - m)e_{m-1} \\ \rho_n(F)e_m &= e_{m+1} \end{aligned}$$

where $e_0, \ldots, e_{n-1}$ are the standard basis vecotrs of $\mathbb{C}^n$ and we have set $e_{-1} = 0$ and $e_n = 0$.

Proof. We must first check that $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a homomorphism of Lie algebras. We have that

$$\begin{aligned} \rho_n([H, E])e_m &= \rho_n(2E)e_m = 2m(n - m)e_{m-1} \\ [\rho_n(H), \rho_n(E)]e_m &= \rho_n(H)\rho_n(E)e_m - \rho_n(E)\rho_n(H)e_m \\ &= m(n - m)\rho_n(H)e_{m-1} - (n - 1 - 2m)\rho_n(E)e_m \\ &= m(n - m)(n + 1 - 2m)e_{m-1} - m(n - 1 - 2m)(n - m)e_{m-1} \\ &= m(n - m)(n + 1 - 2m - n + 1 + 2m)e_{m-1} \\ &= 2m(n - m)e_{m-1} \end{aligned}$$

and so $\rho_n([H, E]) = [\rho_n(H), \rho_n(E)]$. We can similarly check for the other commutation relations to see that ρ_n is a homomorphism of Lie algebras.

We now show that ρ_n is an irreducible representation of $\mathfrak{g}$. Let $W \subseteq V_n$ be a non-zero invariant subspace. Then $\rho_n(H)|_W$ has an eigenvector $0 \neq v \in W$. But then v is also an eigenvector of $\rho_n(H)$ since the e_m are a basis of V_n consisting of $\rho_n(H)$-eigenvectors with distinct eigenvalues, $v = e_m$ for some $0 \leq m \leq n-1$. Since W is invariant, it contains $v = e_m, E^l \cdot e_m, F^k \cdot e_m$ for all l, k and so it contains the whole basis. We must therefore have that $W = V_n$. □

Proposition 9.4. *With notation as above, any finite-dimensional irreducible representation V of $\mathfrak{sl}(2, \mathbb{C})$ is equivalent to one of the V_n.*

Proof. We know that V contains $0 \neq v_0$ such that $Ev_0 = 0$, $H \cdot v_0 = \lambda_0 v_0$ for some $\lambda_0 \in \mathbb{Z}$ and that the sequence $v_0, v_1 = F \cdot v_0, \ldots, v_k = F^k \cdot v_0$ must terminate. The $v_0, v_1, \ldots, v_k$ span an invariant subspace of V and are all linearly independent. Hence they must be a basis for V. We can then define a linear map

$$\begin{aligned} \varphi : V &\to \mathbb{C}^{\dim V} \cong V_{\dim V} \\ v_l &\mapsto e_l \end{aligned}$$

This is an injective intertwiner so Schur's Lemma implies that φ is an isomorphism. □

10 Cartan Subalgebras and Root Space Decomposition

Definition 10.1. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra. A sub-Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a **Cartan subalgebra** of $\mathfrak{g}$ if

1. $H \in \mathfrak{h}$ implies that ad_H is diagonalisable.
2. $\mathfrak{h} = C(h) = \{ x \in \mathfrak{g} \mid [x, H] = 0 \, \forall \, H \in \mathfrak{h} \}$. In other words, $\mathfrak{h}$ is equal to its own **centraliser**.

Remark. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Clearly, the second condition implies that $\mathfrak{h}$ is abelian. This then implies that ad_H are all simultaneously diagonalisable.

Theorem 10.2. *Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra. Then $\mathfrak{g}$ has a non-trivial Cartan subalgebra $\mathfrak{h}$ and such a Cartan subalgebra is unique up to isomorphism.*

Proof. Omitted. □

Definition 10.3. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Since ad_H are simultaneously diagonalisable for all $H \in \mathfrak{h}$, they have a common eigenvector, say $x \in \mathfrak{g}$. Then there exists $\alpha \in \mathfrak{h}^*$ such that

$$\mathrm{ad}_H(x) = \alpha(H)x$$

for all $H \in \mathfrak{h}$. We say that α is a **root** of $\mathfrak{g}$ if $\alpha \neq 0$ and

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [H, x] = \alpha(H)x \, \forall \, H \in \mathfrak{h} \}$$

is not $\{ 0 \}$. In this case, we refer to $\mathfrak{g}_\alpha$ as a **root space**. The collection of all roots of $\mathfrak{g}$ is referred to as a **root system** of $\mathfrak{g}$ and is denoted $\Phi(\mathfrak{g}, \mathfrak{h})$.

Remark. We may decompose $\mathfrak{g}$ into simultaneous eigenspaces of elements of $\mathfrak{h}$ to get the **root space decomposition** of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Example 10.4. Consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) = \{ A \in \mathrm{Mat}(n, \mathbb{C}) \mid \mathrm{Tr}\, A = 0 \}$. We first claim that

$$\mathfrak{h} = \left\{ A = \mathrm{diag}(x_{11}, \ldots, x_{nn}) \;\middle|\; \sum_{i=1}^{n} x_{ii} = 0, x_{ii} \in \mathbb{C} \right\}$$

is a Cartan subalgebra of $\mathfrak{g}$. Consider the basis of matrix units $(E_{ij})_{mn} = \delta_{mi}\delta_{nj}$. Then given $H \in \mathfrak{h}$ we have

$$[H, E_{ij}] = HE_{ij} - E_{ij}H = x_{ii}E_{ij} - x_{jj}E_{ij} = (x_{ii} - x_{jj})E_{ij}$$

and so ad_H is diagonal with respect to this basis. $\mathfrak{h}$ is clearly abelian so it suffices to show that $C(\mathfrak{h}) \subseteq \mathfrak{h}$. Note that a basis for $\mathfrak{g}$ is given by the union of basis elements for $\mathfrak{h}$ and the matrix units E_{ij}. Recall that

$$C(\mathfrak{h}) = \{ A \in \mathfrak{g} \mid [H, x] = 0 \;\forall\, H \in \mathfrak{h} \}$$

Since $[H, E_{ij}] = (x_{ii} - x_{jj})E_{ij}$, we can always find a diagonal matrix $H \in \mathfrak{h}$ such that $[H, E_{ij}] = 0$ for $i \neq j$. Hence the matrix units $E_{ij} \notin C(\mathfrak{h})$. Thus the only possible basis elements in $C(\mathfrak{h})$ are the ones of $\mathfrak{h}$ whence $C(\mathfrak{h}) \subseteq \mathfrak{h}$. Hence $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Now define $e_i \in \mathfrak{h}^*$ by $e_i(H) = x_{ii}$. Then $\mathrm{ad}_H(E_{ij}) = (e_i - e_j)(H)E_{ij}$ and so the roots of $\mathfrak{g}$ are

$$\Phi = \{ e_i - e_j \mid i \neq j \}$$

We can thus find the root spaces:

$$\mathfrak{g}_{e_i - e_j} = \{ x \in \mathfrak{g} \mid [H, x] = (e_i - e_j)(H)x \;\forall\, H \in \mathfrak{h} \} = \mathbb{C}E_{ij}$$

Proposition 10.5. *Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra so that*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Then

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
2. *If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ then $\kappa(x, y) = 0$ unless $\alpha + \beta = 0$ for all $\alpha, \beta \in \{ 0 \} \cup \Phi$.*
3. $\mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}(\Phi)$.

Proof.

Part 1: Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. Then for all $H \in \mathfrak{h}$ we have

$$\begin{aligned}
[H, [x, y]] &= -[y, [H, x]] - [x, [y, H]] \\
&= -[y, [H, x]] + [x, [H, y]] \\
&= -[y, \alpha(H)x] + [x, \beta(H)y] \\
&= -\alpha(H)[y, x] + \beta(H)[x, y] \\
&= \alpha(H)[x, y] + \beta(H)[x, y] \\
&= (\alpha(H) + \beta(H))[x, y]
\end{aligned}$$

and so $[x, y] \in \mathfrak{g}_{\alpha+\beta}$ whence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Part 2: First suppose that $\alpha+\beta \neq 0$. Then there exists $H \in \mathfrak{h}$ such that $\alpha(H)+\beta(H) \neq 0$. Then

$$\begin{aligned}(\alpha(H)+\beta(H))\kappa(x, y) = \kappa(\alpha(H)x, y) + \kappa(x, \beta(H), y) &= \kappa([H, x], y) + \kappa(x, [H, y]) \\ &= -\kappa([x, H], y) + \kappa(x, [H, y]) \\ &= -\kappa(x, [H, y]) + \kappa(x, [H, y]) = 0\end{aligned}$$

where we have used Part 1 and the fact that κ is non-degenerate. Now suppose that $\alpha+\beta = 0$. We want to show that $\kappa(x, y) \neq 0$ for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. But this is equivalent to showing that κ restricted to $\mathfrak{g}_0 = \mathfrak{h}$ is non-degenerate. Let $y \in \mathfrak{h}$. Since $\mathfrak{g}$ is semisimple, Cartan's criterion implies that κ is non-degenerate. Then there exists $\mathfrak{x} \in \mathfrak{g}$ such that $\kappa(x, y) \neq 0$. Now write

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha$$

for some $x_0 \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$. By Part 2, $\kappa(x, y) = \kappa(x_0, y)$. Hence for all $y \in \mathfrak{h}$, we can find an $x_0 \in \mathfrak{h}$ such that $\kappa(x_0, y) \neq 0$.

Part 3: Assume $\operatorname{span}_{\mathbb{C}}(\Phi) \subsetneq \mathfrak{h}^*$. We first claim that there exists non-zero $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ for all $\alpha \in \Phi$. Indeed let $\{ v_i \}$ be a basis for $\mathfrak{h}$. Then there exists a canonical dual basis for $\mathfrak{h}^*$, say $\{ v_i^* \}$ such that $v_i^*(v_j) = \delta_{ij}$. Since $\operatorname{span}_{\mathbb{C}}(\Phi)$ is a proper subspace of $\mathfrak{h}^*$, it must lack at least one of these canonical dual basis elements, say v_i^*. Then, clearly, v_i is a non-zero element of $\mathfrak{h}$ such that $\alpha(v_i) = 0$ for all $\alpha \in H$. It then follows that v_i commutes with all of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ as well as $\mathfrak{h}$ and so v_i commutes with all of $\mathfrak{g}$. Hence $v_i \in Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple and so $Z(g) = \{ 0 \}$ whence $v_i = 0$. But this is a contradiction since v_i is a basis element of $\mathfrak{h}$. We must therefore have that $\operatorname{span}_{\mathbb{C}}(\Phi) = \mathfrak{h}^*$.

□